Linearisable Hierarchies of Evolution Equations in (1+1) Dimensions

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Abstract

In our article [5], *"A tree of linearisable second-order evolution equations by generalised hodograph transformations* [J. Nonlin. Math. Phys. **8** (2001), 342-362] we presented a tree of linearisable (C-integrable) second-order evolution equations in (1+1) dimensions. Expanding this result we report here the complete set of recursion operators for this tree and present several linearisable (C-integrable) hierarchies in (1+1) dimensions.

1 Introduction

In [5] we presented a tree of linearisable (that is, C-integrable) second-order evolution equations which can be transformed to linear partial differential equations. The transformation under which the classification was performed in [5] is the so-called *x-generalised hodograph transformation* defined as follows:

$$
{}_n\mathbf{H}: \begin{cases} dX(x,t) = f_1(x,u)dx + f_2(x,u,u_x,u_{xx},\dots,u_{x^{n-1}})dt \\ dT(x,t) = dt \\ U(X,T) = g(x), \end{cases} \tag{1.1}
$$

with $n = 2, 3, \dots$ and

$$
u_t \frac{\partial f_1}{\partial u} = \frac{\partial f_2}{\partial x} + u_x \frac{\partial f_2}{\partial u} + u_{xx} \frac{\partial f_2}{\partial u_x} + \cdots + u_{x^n} \frac{\partial f_2}{\partial u_{x^{n-1}}}. \tag{1.2}
$$

Applying this transforms on an $(1+1)$-dimensional nth-order autonomous evolution equation

$$
U_T = P\left(U, U_X, U_{XX}, \dots, U_{X^n}\right) \tag{1.3}
$$

leads in general to a $(1+1)$-dimensional nth-order x-dependent evolution equation of the form

$$
u_t = Q(x, u, u_x, u_{xx}, \dots, u_{x^n}). \tag{1.4}
$$

Obviously P and Q are related via (1.1) and (1.2).

The prolongations of (1.1) are

$$\begin{aligned} &U_T = -\frac{f_2}{f_1}\frac{dg}{dx}, \qquad U_X = \frac{\dot g}{f_1} \\ &U_{X^n} = \frac{1}{f_1}\left(D_x\left(\frac{1}{f_1}\right)\right)^{n-2} D_x\left(\frac{\dot g}{f_1}\right), \qquad n = 2, 3, \ \ldots, \end{aligned} \tag{1.5}$$

where $\dot g = dg/dx$ and D_x is the total derivative operator

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \cdots$$

with

$$(D_x a)^2 = D_x(aD_x a), \quad (D_x a)^3 = D_x(aD_x(aD_x a)), \ \ldots$$

Following (1.5), the relation between f_1 and f_2 for (1.3) is

$$f_2(x, u, u_x, \ldots, u_{x^{n-1}}) = -\frac{f_1}{\dot g}\left[P(U, U_X, \ldots U_{X^n})\right]\Big|_{\Omega}, \tag{1.6}$$

where

$$\Omega = \left\{U = g(x),\ U_X = \frac{\dot g}{f_1},\ \ldots,\ U_{X^n} = \frac{1}{f_1}\left(D_x\left(\frac{1}{f_1}\right)\right)^{n-2} D_x\left(\frac{\dot g}{f_1}\right)\right\} \tag{1.7}$$

Equation (1.4) then follows from condition (1.2).

It should be pointed out that the x-generalised hodograph transformation (1.1) is a generalisation of the *extended hodograph transformation* introduced in [4], namely

$$\begin{aligned} &X(x,t) = \int^x f(u(\xi,t))d\xi \\ &T(x,t) = t \\ &U(X,T) = x. \end{aligned} \tag{1.8}$$

For second-order evolution equations we showed [5] that

The most general $(1+1)$*-dimensional second-order evolution equation which may be constructed to be linearisable in*

$$U_T = U_{XX} + \lambda_1 U_X + \lambda_2 U, \qquad \lambda_1, \lambda_2 \in \Re \tag{1.9}$$

via the x*-generalised hodograph transformation (1.1) is necessarily of the form*

$$u_t = F_1(x,u)u_{xx} + F_2(x,u)u_x + F_3(x,u)u_x^2 + F_4(x,u). \tag{1.10}$$

This led to sixteen linearisable second-order evolution equations, eight of which are autonomous (by autonomous equations we mean equations which do not depend explicitly

on their independent variables x and t). These equations are listed in [5] together with their linearising transformations. Only one equation of this class is *autohodograph invariant*, i.e., invariant under an x-generalised hodograph transformation, namely the equation

$$u_t = h(u)u_{xx} + \{h\}_u \, u_x^2, \tag{1.11}$$

where $h \in C^2(\Re)$, $dh/du \neq 0$ and

$$\{h\}_u := -\frac{1}{2}\frac{dh}{du} + h\left(\frac{dh}{du}\right)^{-1}\frac{d^2h}{du^2}. \tag{1.12}$$

One can show the following:

The most general nth-order evolution equation which is linearisable in

$$U_T = \lambda_0 U + \sum_{l=1}^{n} \lambda_l U_{X^l}, \qquad \lambda_j \in \Re \tag{1.13}$$

by repeatedly applying the transformation (1.1), is necessarily of the form

$$u_t = \sum_{r=1}^{n} \sum_{k_1,k_2,\dots k_r=0}^{r} F_{k_1k_2\dots k_r}(x,u)u_x^{k_1}u_{xx}^{k_2}\cdots u_{x^r}^{k_r} + F_0(x,u), \tag{1.14}$$

where $F_{k_1k_2\dots k_r} \in C^2(\Re)$ are functions of x and u, and

$$\sum_{j=1}^{r} jk_j = r, \quad k_l \in \{0,1,\dots,n\}, \qquad 1 \le l \le r, \qquad 1 \le r \le n.$$

The objective in the present paper is to construct higher-order linearisable autonomous evolution equations by the use of the tree of second-order linearisable equations gven in [5]. This is achieved by calculating the recursion operators for the second-order equations. The hierarchies obtained by the recursion operators are all linearisable via the x-generalised hodograph transformations and are of the general quasi-linear form (1.14). We do, however, restrict ourselves to the autonomous case. We furthermore write the equations in potential form and use the pure hodograph transformation and corresponding recursion operators to construct hierarchies of autonomous nonlinear evolution equations and their linearising transformations.

We point out that the x-generalised hodograph transformation can be applied only on autonomous evolution equations and may produce nonautonomous evolution equation of the same order. However, it is easy to show that any nonautonomous evolution equation which has been generated by such a transformation can always be made autonomous. This is achieved by writing the obtained equation in potential form followed by the pure hodograph transformation.

2 Second-order linearisable equations and their potential forms

The following eight second-order evolution equations were constructed to be linearisable via the x-generalised hodograph transformation (1.1) [5]:

$$u_t = h_1 u_{xx} + \{h_1\}_u u_x^2, \qquad \dot h_1(u) \neq 0 \qquad \text{(I)}$$
$$u_t = h_2 u_{xx} + \lambda h_2 u_x + \{h_2\}_u u_x^2, \qquad \dot h_2(u) \neq 0,\ \lambda \neq 0 \qquad \text{(II)}$$
$$u_t = h_3 u_{xx} + \{h_3\}_u u_x^2 + 2\lambda_2 h^{3/2} \dot h_3^{-1}, \qquad \dot h_3(u) \neq 0,\ \lambda_2 \neq 0 \qquad \text{(III)}$$
$$u_t = u_{xx} + \lambda_4 u_x + h_4^{-1}(\lambda_2 - \dot h_4) u_x^2 + h_4, \qquad h_4(u) \neq 0,\ \lambda_2 \neq 0 \qquad \text{(IV.1)}$$
$$u_t = u_{xx} + \lambda_4 u_x - h_4^{-1} \dot h_4 u_x^2 + h_4, \qquad h_4(u) \neq 0 \qquad \text{(IV.2)}$$
$$u_t = h_5 u_{xx} + \left(\lambda h_5 - \lambda_2 \lambda^{-1}\right) u_x + \{h_5\}_u u_x^2, \qquad \dot h_5(u) \neq 0,\ \lambda \neq 0,\ \lambda_2 \neq 0 \qquad \text{(V)}$$
$$u_t = u_{xx} + h_6 u_x + \ddot h_6 \dot h_6^{-1} u_x^2, \qquad h_6(u) \neq 0 \qquad \text{(VI)}$$
$$u_t = h_7 u_{xx} + \lambda_3 u_x + \{h_7\}_u u_x^2, \qquad \dot h_7(u) \neq 0,\ \lambda_3 \neq 0 \qquad \text{(VII)}$$
$$u_t = u_{xx} + \lambda_8 u_x + h_8 u_x^2 \qquad \text{(VIII)}$$

Here h_j are arbitrary C^2-functions depending on u (with the indicated restrictions). The bracket $\{h_j\}_u$ is defined by (1.12). All λ's are arbitrary constants, unless otherwise stated. Here and in the rest of this paper the overdot on h_j denotes differentiation with respect to u.

The equations (I - VIII) listed above may be linearised to (1.9), i.e.,

$$U_T = U_{XX} + \lambda_1 U_X + \lambda_2 U - \lambda_3, \qquad \lambda_j \in \Re$$

by the following transformations, respectively [5]:

(Trans-I) $\quad x = U(X,T), \quad dt = dT, \quad h_1(u) = U_X^2.$

(Trans-II) $\quad x = \dfrac{1}{\lambda} \ln|\lambda U|, \quad dt = dT, \quad h_2(u) = \dfrac{1}{\lambda^2}\left(\dfrac{U_X}{U}\right)^2.$

(Trans-III) $\quad x = \dfrac{2}{\lambda_2}\left(\dfrac{U_X}{U}\right), \quad dt = dT, \quad h_3(u) = \dfrac{4}{\lambda_2^2}\left[\dfrac{\partial}{\partial X}\left(\dfrac{U_X}{U}\right)\right]^2.$

(Trans-IV.1) $\quad \lambda_2 \neq 0: \quad dx = dX, \quad dt = dT, \quad \displaystyle\int^u \frac{1}{h_4(\xi)} d\xi = \frac{1}{\lambda_2} \ln|\lambda U|.$

(Trans-IV.2) $\quad \lambda_2 = 0: \quad dx = dX, \quad dt = dT, \quad \dfrac{u_x}{h_4(u)} = -\dfrac{1}{\lambda_3} U.$

(Trans-V) $\quad x = \dfrac{1}{\lambda} \ln|\lambda U|, \quad dt = dT, \quad h_5(u) = \dfrac{1}{\lambda^2}\left(\dfrac{U_X}{U}\right)^2.$

(Trans-VI) $\quad dx = dX, \quad dt = dT, \quad h_6(u) = 2\dfrac{U_X}{U}.$

(Trans-VII) $\quad x = U, \quad dt = dT, \quad h_7(u) = U_X^2.$

$$\text{(Trans-VIII)} \quad dx = dX, \quad dt = dT, \quad \int^u \exp\left(\int^\xi h_8(\xi')d\xi'\right) d\xi = U_X.$$

We now write equations (I) - (VIII) in potential form and introduce new arbitrary functions $\phi_j(x)$, which then lead to new linearisable autonomous evolution equations after transforming the corresponding potential equations by the pure hodograph transformation. The linearising transformations for the equations so obtained result by composing the potentials and hodograph tranformations with the corresponding transformations (Trans-I)–(Trans-VIII) listed above.

Equation (I): We give two possibilities for writing (I) in potential form:
(I.i): Let

$$\begin{aligned} v_x(x,t) &= h_1^{-1/2}(u) + \phi_1(x) \\ v_t(x,t) &= -\frac{1}{2} h_1^{-1/2}(u)\dot{h}_1(u)u_x - \lambda_1, \end{aligned}$$

where ϕ_1 is an arbitrary function of x and $\lambda_1 \in \Re$. Then $v_{xt} = v_{tx}$ leads to (I). The potential equation takes the form

$$v_t = \frac{v_{xx} - \phi_{1x}(x)}{(v_x - \phi_1(x))^2} - \lambda_1. \tag{2.1}$$

Transforming (2.1) by the pure hodograph transformation

$$v(x,t) = \chi, \quad t = \tau, \quad x = V(\chi,\tau) \tag{2.2}$$

leads to the autonomous equation

$$V_\tau = \frac{V_{\chi\chi} + \phi_1'(V)V_\chi^3}{(1 - \phi_1(V)V_\chi)^2} + \lambda_1 V_\chi, \tag{2.3}$$

where $\phi' = d\phi_1/dV$. By the given change of variables and (Trans-I), it follows that (2.3) linearises to

$$U_T = U_{XX} + \lambda_1 U_X$$

by the transformation

$$V(\chi,\tau) = U(X,T), \quad d\tau = dT, \quad V_\chi^{-1} - \phi_1(V) = U_X^{-1}. \tag{2.4}$$

(I.ii): Let

$$\begin{aligned} v_x(x,t) &= x h_1^{-1/2}(u) + \phi_1(x) \\ v_t(x,t) &= h_1^{1/2}(u) - \frac{x}{2} h_1^{-1/2}(u)\dot{h}_1(u)u_x - \lambda_1. \end{aligned}$$

Here ϕ_1 is an arbitrary function of x and λ_1 is an arbitrary constant. Clearly $v_{xt} = v_{tx}$ leads to (I). The potential equation takes the form

$$v_t = \frac{v_{xx} - \phi_{1x}(x)}{(v_x - \phi_1(x))^2} x^2 - \lambda_1. \tag{2.5}$$

Transforming (2.5) by the pure hodograph transformation (2.2) leads to the autonomous equation

$$V_\tau = \frac{V_{\chi\chi} + \phi_1'(V)V_\chi^3}{(1-\phi_1(V)V_\chi)^2} V + \lambda_1 V_\chi, \tag{2.6}$$

where $\phi' = d\phi_1/dV$. By the given change of variables and (Trans-I), it follows that (2.6) linearises to

$$U_T = U_{XX} + \lambda_1 U_X$$

by the transformation

$$V(\chi,\tau) = U(X,T), \quad d\tau = dT, \quad V_\chi^{-1} - \phi_1(V) = UU_X^{-1}. \tag{2.7}$$

The same procedure can be followed for equations (II) - (VIII). We list the results below:

Equation (II): Two cases are given.
(II.i): Let

$$\begin{aligned} v_x(x,t) &= h_2^{-1/2}(u) + \phi_2(x) \\ v_t(x,t) &= -\frac{1}{2} h_2^{-1/2}(u)\dot{h}_2(u)u_x - \lambda h_2^{1/2}(u) - \lambda_1, \quad \lambda \neq 0. \end{aligned}$$

The potential equation is then

$$v_t = \frac{v_{xx} - \phi_{2x}(x)}{(v_x - \phi_2(x))^2} - \frac{\lambda}{v_x - \phi_2} - \lambda_1. \tag{2.8}$$

By the pure hodograph transformation (2.2), (2.8) leads to

$$V_\tau = \frac{V_{\chi\chi} + \phi_2'(V)V_\chi^3}{(1-\phi_2(V)V_\chi)^2} + \frac{\lambda V_\chi^2}{1-\phi_2(V)V_\chi} + \lambda_1 V_\chi, \tag{2.9}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X$$

by the transformation

$$V(\chi,\tau) = \frac{1}{\lambda}\ln|U(X,T)|, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_2(V) = \lambda UU_X^{-1}. \tag{2.10}$$

(II.ii): Let

$$\begin{aligned} v_x(x,t) &= e^{\lambda x} h_2^{-1/2}(u) + \phi_2(x) \\ v_t(x,t) &= -\frac{1}{2} e^{\lambda x} h_2^{-1/2}(u)\dot{h}_2(u)u_x - \lambda_1, \quad \lambda \neq 0. \end{aligned}$$

The potential equation is then

$$v_t = e^{2\lambda x}\frac{v_{xx} - \phi_{2x}(x)}{(v_x - \phi_2(x))^2} - \frac{\lambda e^{2\lambda x}}{v_x - \phi_2(x)} - \lambda_1. \tag{2.11}$$

By the pure hodograph transformation (2.2), (2.11) leads to

$$V_\tau = e^{2\lambda V}\frac{V_{\chi\chi} + \phi_2'(V)V_\chi^3}{(1 - \phi_2(V)V_\chi)^2} + e^{2\lambda V}\frac{\lambda V_\chi^2}{1 - \phi_2(V)V_\chi} + \lambda_1 V_\chi, \tag{2.12}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X$$

by the transformation

$$V(\chi,\tau) = \frac{1}{\lambda}\ln|U(X,T)|, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_2(V) = \lambda U^2 U_X^{-1}. \tag{2.13}$$

Equation (III): Two cases are given.
(III.i): Let

$$\begin{aligned} v_x(x,t) &= h_3^{-1/2}(u) + \phi_3(x) \\ v_t(x,t) &= -\frac{1}{2}h_3^{-1/2}(u)\dot{h}_3(u)u_x - \lambda_2 x - \lambda_1, \quad \lambda_2 \neq 0. \end{aligned}$$

The potential equation is then

$$v_t = \frac{v_{xx} - \phi_{3x}(x)}{(v_x - \phi_3(x))^2} - \lambda_2 x - \lambda_1. \tag{2.14}$$

By the pure hodograph transformation (2.2), (2.14) leads to

$$V_\tau = \frac{V_{\chi\chi} + \phi_3'(V)V_\chi^3}{(1 - \phi_3(V)V_\chi)^2} + \lambda_2 V V_\chi + \lambda_1 V_\chi, \tag{2.15}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X$$

by the transformation

$$V(\chi,\tau) = \frac{2}{\lambda_2}\frac{U_X}{U}, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_3(V) = \frac{\lambda_2}{2}\left(\frac{\partial}{\partial X}\left(\frac{U_X}{U}\right)\right)^{-1}. \tag{2.16}$$

(III.ii): Let

$$\begin{aligned} v_x(x,t) &= x h_3^{-1/2}(u) + \phi_3(x) \\ v_t(x,t) &= h_3^{1/2}(u) - \frac{1}{2}x h_3^{-1/2}(u)\dot{h}_3(u)u_x - \frac{\lambda_2 x^2}{2} - \lambda_1, \quad \lambda \neq 0, \ \lambda_2 \neq 0. \end{aligned}$$

The potential equation then takes the form

$$v_t = \left(\frac{v_{xx} - \phi_{3x}(x)}{(v_x - \phi_3(x))^2}\right) x^2 - \frac{\lambda_2 x^2}{2} - \lambda_1. \tag{2.17}$$

By the pure hodograph transformation (2.2), (2.17) leads to

$$V_\tau = V^2 \left(\frac{V_{\chi\chi} + \phi_3'(V) V_\chi^3}{(1 - \phi_3(V) V_\chi)^2}\right) + \frac{\lambda_2}{2} V^2 V_\chi + \lambda_1 V_\chi, \tag{2.18}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X + \lambda_2 U$$

by the transformation

$$V(\chi, \tau) = \frac{2}{\lambda_2}\frac{U_X}{U}, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_3(V) = \frac{U_X}{U}\left(\frac{\partial}{\partial X}\left(\frac{U_X}{U}\right)\right)^{-1}. \tag{2.19}$$

Equation (IV.1): Let

$$v_x(x,t) = \exp\left(\lambda_2 \int^u \frac{1}{h_4(\xi)} d\xi + rx\right) + \phi_4(x)$$

$$v_t(x,t) = \left(\frac{\lambda_2}{h_4(u)} u_x + \frac{\lambda_2}{r}\right) \exp\left(\lambda_2 \int^u \frac{1}{h_4(\xi)} d\xi + rx\right) - \lambda_1, \qquad \lambda_2 \neq 0,$$

where

$$r = \frac{\lambda_1}{2} \pm \left(\frac{\lambda_1^2}{4} - \lambda_2\right)^{1/2}. \tag{2.20}$$

The potential equation then becomes

$$v_t = v_{xx} - \phi_{4x}(x) + (\lambda_1 - 2r)(v_x - \phi_4(x)) - \lambda_1. \tag{2.21}$$

By the pure hodograph transformation (2.2), (2.21) leads to

$$V_\tau = V_\chi^{-2} V_{\chi\chi} + \phi_4'(V) V_\chi + (2r - \lambda_1)(1 - \phi_4(V) V_\chi) + \lambda_1 V_\chi \tag{2.22}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X + \lambda_2 U$$

by the transformation

$$V(\chi, \tau) = X, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_4(V) = e^{rX} U, \tag{2.23}$$

where r is given by (2.20).

Equation (IV.2): Let

$$v_x(x,t) = e^{\lambda_1 x}\int^u \frac{1}{h_4(\xi)}d\xi + \phi_4(x)$$
$$v_t(x,t) = e^{\lambda_1 x}\frac{u_x}{h_4(u)} + \frac{1}{\lambda_1}e^{\lambda_1 x} - \lambda_1, \quad \lambda_1 \neq 0.$$

The potential equation then becomes

$$v_t = v_{xx} - \lambda_1(v_x - \phi_4(x) - \phi_{4x}(x)) + \frac{1}{\lambda_1}e^{\lambda_1 x} - \lambda_1. \tag{2.24}$$

By the pure hodograph transformation (2.2), (2.24) leads to

$$V_\tau = V_\chi^{-2}V_{\chi\chi} + \phi_4'(V)V_\chi + \lambda_1(1 - \phi_4(V)V_\chi) - \frac{1}{\lambda_1}e^{\lambda_1 V}V_\chi + \lambda_1 V_\chi \tag{2.25}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X$$

by the transformation

$$V(\chi,\tau) = X, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_4(V) = e^{\lambda_1 X}\int^X U(\xi,T)d\xi. \tag{2.26}$$

Equation (V): Let

$$v_x(x,t) = h_5^{-1/2}(u) + \phi_5(x)$$
$$v_t(x,t) = -\frac{1}{2}h_5^{-1/2}\dot{h}_5(u)u_x - \lambda h_5^{1/2}(u) - \frac{\lambda_2}{\lambda}h_5^{-1/2}(u) - \lambda_1, \quad \lambda \neq 0.$$

The potential equation then becomes

$$v_t = \frac{v_{xx} - \phi_{5x}(x)}{(v_x - \phi_5(x))^2} - \frac{\lambda}{v_x - \phi_5(x)} - \frac{\lambda_2}{\lambda}(v_x - \phi_5(x)) - \lambda_1. \tag{2.27}$$

By the pure hodograph transformation (2.2), (2.27) leads to

$$V_\tau = \frac{V_{\chi\chi} + \phi_5'(V)V_\chi^3}{(1 - \phi_5(V)V_\chi)^2} + \frac{\lambda V_\chi^2}{1 - \phi_5(V)V_\chi} + \frac{\lambda_2}{\lambda}(1 - \phi_5(V)V_\chi) + \lambda_1 V\chi \tag{2.28}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X + \lambda_2 U$$

by the transformation

$$V(\chi,\tau) = \frac{1}{\lambda}\ln|\lambda U(X,T)|, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_5(V) = \lambda\frac{U}{U_X}. \tag{2.29}$$

Equation (VI): Let

$$v_x(x,t) = h_6(u) + \phi_6(x)$$
$$v_t(x,t) = \dot{h}_6(u)u_x + \frac{1}{2}h_6^2(u).$$

The potential equation then takes the form

$$v_t = v_{xx} - \phi_6(x) + \frac{1}{2}(v_x - \phi_6(x))^2. \tag{2.30}$$

By the pure hodograph transformation (2.2), (2.30) leads to

$$V_\tau = V_\chi^{-2}V_{\chi\chi} + \phi_6'(V)V_\chi - \frac{1}{2}(1 - \phi_6(V)V_\chi)^2 V_\chi^{-1} \tag{2.31}$$

which linearises to

$$U_T = U_{XX} \tag{2.32}$$

by the transformation

$$V(\chi,\tau) = X, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_6(V) = 2\frac{U_X}{U}. \tag{2.33}$$

Equation (VII): Let

$$v_x(x,t) = h_7^{-1/2}(u) + \phi_7(x)$$
$$v_t(x,t) = -\frac{1}{2}h_7^{-1/2}\dot{h}_7(u)u_x - \lambda_3 h_7^{-1/2}(u) - \lambda_1.$$

The potential equation then becomes

$$v_t = \frac{v_{xx} - \phi_{7x}(x)}{(v_x - \phi_7(x))^2} - \lambda_3(v_x - \phi_7(x)) - \lambda_1. \tag{2.34}$$

By the pure hodograph transformation (2.2), (2.34) leads to

$$V_\tau = \frac{V_{\chi\chi} + \phi_7'(V)V_\chi^3}{(1 - \phi_7(V)V_\chi)^2} + \lambda_3(1 - \phi_7(V)V_\chi) + \lambda_1 V\chi \tag{2.35}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X + \lambda_3$$

by the transformation

$$V(\chi,\tau) = U(X,T), \quad d\tau = dT, \quad V_\chi^{-1} - \phi_7(V) = U_X^{-1}. \tag{2.36}$$

Equation (VIII): Let

$$v_x(x,t) = e^{\lambda_1 x}\int^u \exp\left[\int^\xi h_8(\xi')d\xi'\right]d\xi + \phi_8(x)$$

$$v_t(x,t) = e^{\lambda_1 x}\exp\left[\int^u h_8(\xi)d\xi\right] u_x - \lambda_1.$$

The potential equation then takes the form

$$v_t = v_{xx} - \lambda_1(v_x - \phi_8(x)) - \phi_{8x}(x) - \lambda_1 \tag{2.37}$$

By the pure hodograph transformation (2.37) leads to

$$V_\tau = V_\chi^{-2} V_{\chi\chi} + \phi_8'(V)V_\chi + \lambda_1(1 - \phi_8(V)V_\chi) + \lambda_1 V_\chi \tag{2.38}$$

which linearises to

$$U_T = U_{XX} + \lambda_1 U_X$$

by the transformation

$$V(\chi,\tau) = X, \quad d\tau = dT, \quad V_\chi^{-1} - \phi_8(V) = e^{\lambda_1 X} U_X. \tag{2.39}$$

The same procedure could now, in principle, be applied again on these new equations (2.3), (2.6), (2.9), (2.12), (2.15), (2.18), (2.22), (2.25), (2.28), (2.31), (2.35), and (2.35), to construct new chains of linearisable equations. We present here only one example.

An example to further extend (2.3): Let

$$W_\chi(\chi,\tau) = V + \Omega(\chi)$$
$$W_\tau(\chi,\tau) = \frac{V\chi}{(1 - \phi_1(V)V_\chi)},$$

where Ω is an arbitrary function of χ. We set $\lambda_1 = 0$ in (2.3). The potential equation then takes the form

$$W_\tau = \frac{W_{\chi\chi} - \Omega_\chi}{1 - (W_{\chi\chi} - \Omega_\chi)\phi_1(W_\chi - \Omega)}. \tag{2.40}$$

Performing the pure hodograph transformation

$$W(\chi,\tau) = \xi, \quad \chi = \omega(\xi,\eta), \quad \tau = \eta$$

on (2.40) leads to

$$\omega_\eta = \frac{\omega_\xi \omega_{\xi\xi} + \omega_\xi^4 \Omega_\omega}{\omega_\xi^3 + (\omega_{\xi\xi} + \omega_\xi^3 \Omega_\omega)\phi_1(\omega_\xi^{-1} - \Omega)} \tag{2.41}$$

which may be linearised to

$$U_X = U_{XX}$$

by the transformation

$$\begin{aligned} &\omega_\xi^{-1} - \Omega(\omega) = U(X,T) \\ &d\eta = dT \\ &\omega_\xi^{-3}\omega_{\xi\xi} + \Omega'(\omega) = \frac{U_X}{U_X \phi_1(U) - 1} \end{aligned} \tag{2.42}$$

where $\Omega' = d\Omega/d\omega$.

A detailed analysis of this type of extensions will be considered elsewhere.

3 Recursion operators and hierarchies of linearisable equations

In this section we give the recursion operators for the linearisable equations listed in Section 2. Before we list our results, we mention some relevant facts concerning recursion operators of evolution equations.

A recursion operator in two independent variables x and t is a linear integro-differential operator of the form

$$R[u] = \sum_{j=0}^{l} P_j D_x^j + \sum_{j=1}^{s} Q_j D_x^{-j}, \tag{3.1}$$

where D_x denotes the total x-derivative and D_x^{-j} the j-fold product of the inverse of D_x. The coefficients P_j and Q_j depend in general on x, t and u, as well as a finite number of x derivatives of u. These operators were introduced by Olver [10] to generate (infinite) sequences of Lie-Bäcklund (also called Generalised) symmetry generators. For an nth-order evolution equation in u,

$$u_t = F(x, t, u, u_x, u_{xx}, \ldots, u_{x^n}), \tag{3.2}$$

a Lie-Bäcklund symmetry generator Z is of the form

$$Z = \eta(x, t, u, u_x, u_{xx}, \ldots, u_{x^q}) \frac{\partial}{\partial u}, \qquad q > n, \tag{3.3}$$

if it exists. The recursion operator is such that when acting on the Lie-Bäcklund symmetry generator, the result is still a Lie-Bäcklund symmetry of the same evolution equation (3.2), i.e.,

$$\eta_{i+1} = R[u]\eta_i. \tag{3.4}$$

The recursion operator then satisfy the commutation relation

$$[L[u]\, ,\, R[u]] = D_t R[u], \tag{3.5}$$

where $L[u]$ is the linear operator

$$L[u] = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial u_x} D_x + \frac{\partial F}{\partial u_{xx}} D_x^2 + \cdots + \frac{\partial F}{\partial u_{x^n}} D_x^n \tag{3.6}$$

and $D_t R[u]$ calculates the explicit derivative with respect to t. For more details we refer to [11, 2, 8].

In this paper we consider only autonomous evolution equations, i.e., equations invariant under translation in t and x. Therefore the equations admit the point symmetry generators

$$u_t \frac{\partial}{\partial u}, \qquad u_x \frac{\partial}{\partial u}. \tag{3.7}$$

A hierarchy of evolution equations can then be obtained by applying the recursion operator on the t-translation symmetry, or equivalently on F, i.e.,

$$u_t = R^m[u]F, \qquad m \in \mathcal{N}. \tag{3.8}$$

3.1 Recursion operators for (I) - (VIII)

Below we list the recursion operators for the eight linearisable equations (I - VIII) listed in Section 2. Note that equations (II) and (V) share the same recursion operator, as do equations (I) and (VII):

$$\textbf{I} \quad R_1[u] = h_1^{1/2} D_x + \frac{\{h_1\}_u}{h_1^{1/2}} u_x + \frac{1}{2}\left(h_1 u_{xx} + \{h_1\}_u u_x^2\right) D_x^{-1} \frac{\dot{h}_1}{h_1^{3/2}}$$

$$\textbf{II} \quad R_2[u] = h_2^{1/2} D_x + \frac{\{h_2\}_u}{h_2^{1/2}} u_x + \lambda h_2^{1/2} + \frac{1}{2}\left(h_2 u_{xx} + \{h_2\}_u u_x^2 + \lambda h_2 u_x\right) D_x^{-1} \frac{\dot{h}_2}{h_2^{3/2}}$$

$$\textbf{III} \quad R_3[u] = h_3^{1/2} D_x + \frac{\{h_3\}_u}{h_3^{1/2}} u_x + \frac{1}{2}\lambda_2 x + \frac{1}{2}\left(h_3 u_{xx} + \{h_3\}_u u_x^2 + 2\lambda_2 \frac{h_3^{3/2}}{\dot{h}_3}\right) D_x^{-1} \frac{\dot{h}_3}{h_3^{3/2}}$$

$$\textbf{IV} \quad R_4[u] = D_x + \frac{1}{h_4}\left(\lambda_2 - \dot{h}_4\right) u_x$$

$$\textbf{V} \quad R_5[u] = h_5^{1/2} D_x + \frac{\{h_5\}_u}{h_5^{1/2}} u_x + \lambda h_5^{1/2} + \frac{1}{2}\left(h_5 u_{xx} + \{h_5\}_u u_x^2 + \lambda h_5 u_x\right) D_x^{-1} \frac{\dot{h}_5}{h_5^{3/2}}$$

$$\textbf{VI} \quad R_6[u] = D_x + \frac{\ddot{h}_6}{\dot{h}_6} u_x + \frac{1}{2} h_6 + \frac{1}{2} u_x D_x^{-1} \dot{h}_6$$

$$\textbf{VII} \quad R_7[u] = h_7^{1/2} D_x + \frac{\{h_7\}_u}{h_7^{1/2}} u_x + \frac{1}{2}\left(h_7 u_{xx} + \{h_7\} u_x^2\right) D_x^{-1} \frac{\dot{h}_7}{h_7^{3/2}}$$

$$\textbf{VIII} \quad R_8[u] = D_x + h_8 u_x$$

The recursion operators R_4 and R_6 are also given in [7].

Below we list the first nontrivial Lie-Bäcklund symmetries of the eight linearisable second-order evolution equations (I - VIII), with the same equation numbers. We note that all symmetries are of order three and only equation (III) has an x-dependent third-order symmetry.

$$\textbf{I} \quad Z_1 = h_1^{3/2}\left(u_{xxx} + 3\frac{\ddot{h}_1}{\dot{h}_1} u_x u_{xx} + \frac{\dddot{h}_1}{\dot{h}_1} u_x^3\right) \frac{\partial}{\partial u}$$

$$\textbf{II} \quad Z_2 = h_2^{3/2}\left(u_{xxx} + 3\frac{\ddot{h}_2}{\dot{h}_2} u_x u_{xx} + \frac{\dddot{h}_2}{\dot{h}_2} u_x^3 + 3\lambda u_{xx} + 3\lambda \frac{\ddot{h}_2}{\dot{h}_2} u_x^2 + 2\lambda^2 u_x\right) \frac{\partial}{\partial u}$$

$$\textbf{III} \quad Z_3 = \left\{ h_3^{3/2}\left(u_{xxx} + 3\frac{\ddot{h}_3}{\dot{h}_3} u_x u_{xx} + \frac{\dddot{h}_3}{\dot{h}_3} u_x^3\right) \right.$$
$$\left. + \frac{3}{2}\lambda_2 x \left(h_3 u_{xx} + \{h_3\} u_x^2 + 2\lambda_2 \frac{h_3^{3/2}}{\dot{h}_3}\right) + 3\lambda_2 h_3 u_x \right\} \frac{\partial}{\partial u}$$

IV $$Z_4=\left\{u_{xxx}+3\frac{1}{h_4}\left(\lambda_2-\dot{h}_4\right)u_xu_{xx}+\frac{1}{h_4^2}\left(\lambda_2^2-3\lambda_2\dot{h}_4+2\dot{h}_4^{\,2}-h\ddot{h}_4\right)u_x^3+\lambda_2u_x\right\}\frac{\partial}{\partial u}$$

V $$Z_5=h^{3/2}\left(u_{xxx}+3\frac{\ddot{h}_5}{\dot{h}_5}u_xu_{xx}+\frac{\dddot{h}_5}{\dot{h}_5}u_x^3+3\lambda u_{xx}+3\lambda\frac{\ddot{h}_5}{\dot{h}_5}u_x^2+2\lambda^2u_x\right)\frac{\partial}{\partial u}$$

VI $$Z_6=\left\{u_{xxx}+3\frac{\ddot{h}_6}{\dot{h}_6}u_xu_{xx}+\frac{\dddot{h}_6}{\dot{h}_6}u_x^3+\frac{3}{2}h_6u_{xx}+\frac{3}{2}\left(\dot{h}_6+h_6\frac{\ddot{h}_6}{\dot{h}_6}\right)u_x^2+\frac{3}{4}h_6^2u_x\right\}\frac{\partial}{\partial u}$$

VII $$Z_7=h_7^{3/2}\left(u_{xxx}+3\frac{\ddot{h}_7}{\dot{h}_7}u_xu_{xx}+\frac{\dddot{h}_7}{\dot{h}_7}u_x^3\right)\frac{\partial}{\partial u}$$

VIII $$Z_8=\left\{u_{xxx}+3h_8u_xu_{xx}+\left(\dot{h}_8+h_8^2\right)u_x^3\right\}\frac{\partial}{\partial u}$$

The hierarchies resulting from the second-order linearisable equations (I) - (VIII) can then be written in the form

$$u_t=\left(R_k^m[u]+\sum_{i=0}^{m-1}R_k^i[u]C_i\right)F_k(u,u_x,u_{xx}) \tag{3.9}$$

where $m=1,2,\ldots$, C_i are arbitrary constants, and F_k denotes the r.h.s of the second-order equations (I) - (VIII). The transformations linearising the hierarchies (3.9) are the same as the transformation (Trans-I) – (Trans-VIII) for the second-order equations (I) - (VIII) listed in Section 2, whereby the linearised equation in U with independent variables X and T are of the same order as the equation from the hierarchy which is being linearised. We'll refer to these linearisable hierarchies as hierarchies (I) - (VIII), corresponding to the second-order equations (I) - (VIII).

As an example we write explicitly the third-order equation from the hierarchy (I) with $h_1=u^n$ $(n\in\mathcal{Q}\backslash\{0\})$ using its recursion operator R_1, which now takes the form

$$R_1[u]=u^{n/2}D_x+\frac{n-2}{2}u^{(n/2)-1}u_x+\frac{n}{2}\left(u^nu_{xx}+\frac{n-2}{2}u^{n-1}u_x^2\right)D_x^{-1}u^{-(n/2)-1}. \tag{3.10}$$

Thus

$$u_t=(R_1[u]+C_0)\left(u^nu_{xx}+\frac{n-2}{2}u^{n-1}u_x^2\right),$$

is the third-order linearisable equation, namely

$$
\begin{aligned}
u_t = & u^{3n/2}u_{xxx} + 3(n-1)u^{(3n/2)-1}u_x u_{xx} + C_0 u^n u_{xx} \\
& + \frac{C_0}{2}(n-2)u_x^2 u^{n-1} + (n-1)(n-2)u^{(3n/2)+1}u_x^3,
\end{aligned}
\tag{3.11}
$$

which linearises to

$$U_T = U_{XXX} + C_0 U_{XX}$$

by the transformation (Trans-I).

3.2 Some examples of hierarchies (I)-(VIII) in the literature

Third-Order equations:

We give two equations from the list in [3] by F Calogero:

(3.138 [3]) $\quad u_t = u^3 u_{xxx} + (2u^3 + au^2)u_x + (3u^3 + au^2)u_{xx} + 3u^2 u_x^2 + 3u^2 u_x u_{xx},$

which is part of hierarchy **II**, namely

$$u_t = (R_2[u] + C_0)\left(h_2 u_{xx} + \lambda h_2 u_x + \{h_2\}_u u_x^2\right),$$

with $h_2 = u^2$, $\lambda = 1$ and $C_0 = a$ (arbitrary constant). This equation linearises to

$$U_T = U_{XXX} + aU_{XX}$$

by the transformation

$$x = \ln U, \quad dt = dT, \quad u(x,t) = U^{-1}U_X.$$

(3.133 [3]) $\quad u_t = u_{xxx} + au_{xx} + 3u_x u_{xx} + au_x^2 + u_x^3,$

which is part of hierarchy **VIII**, namely

$$u_t = (R_8[u] + C_0)\left(u_{xx} + h_8 u_x^2\right),$$

with $h_8 = 1$ and $C_0 = a$ (arbitrary constant). This equation linearises to

$$U_T = U_{XXX} + aU_{XX}$$

by the transformation

$$dx = dX, \quad dt = dT, \quad u(x,t) = \ln|U_X|.$$

An equations introduced by S Kawamoto [9]:

$$u_t = u^3 u_{xxx} + (1+\alpha)u^2 u_{xx}$$

which is part of hierarchy **I**, namely

$$u_t = (R_1[u] + C_0)\left(h_1 u_{xx} + \{h_1\}_u u_x^2\right),$$

with $h_1 = u^2$, $\alpha = 2$ and $C_0 = 0$. This equation linearises to

$$U_T = U_{XXX}$$

by the transformation

$$x = U, \quad dt = dT, \quad u(x,t) = U_X.$$

Fourth-order equations:

An equation introduced by S Kawamoto [9]:

$$u_t = u^4 u_{xxxx} + 6u^3 u_x u_{xxx} + 4u^3 u_{xx}^2 + 7u^2 u_x^2 u_{xx}$$

which is part of hierarchy **I**, namely

$$u_t = \left(R_1^2[u] + R_1[u]C_1 + C_0\right)\left(h_1 u_{xx} + \{h_1\} u_x^2\right),$$

with $h_1 = u^2$ and $C_1 = C_0 = 0$. This equation linearises to

$$U_T = U_{XXXX}$$

by the transformation

$$x = U, \quad dt = dT, \quad u(x,t) = U_X.$$

Fifth-order equations:

An equation by S Kawamoto [9]:

$$\begin{aligned} u_t = {} & u^5 u_{xxxxx} + 10u^4 u_x u_{xxxx} + 15u^4 u_{xx} u_{xxx} + 25u^3 u_x^2 u_{xxx} \\ & + 30u^3 u_x u_{xx}^2 + 15u^2 u_x^3 u_{xx} \end{aligned}$$

which is part of hierarchy **I**, namely

$$u_t = \left(R_1^3[u] + R_1^2[u]C_2 + R_1[u]C_1 + C_0\right)\left(h_1 u_{xx} + \{h_1\} u_x^2\right),$$

with $h_1 = u^2$ and $C_2 = C_1 = C_0 = 0$. This equation linearises to

$$U_T = U_{XXXXX}$$

by the transformation

$$x = U, \quad dt = dT, \quad u(x,t) = U_X.$$

An equation by A H Bilge [1]:

$$u_t = u_{xxxxx} + \beta(u_{xxxx}u_x + 2u_{xxx}u_{xx}) + \beta^2\left(\frac{2}{5}u_{xxx}u_x^2 + \frac{3}{5}u_{xx}^2u_x\right)$$
$$+\frac{2}{25}\beta^3u_{xx}u_x^3 + \frac{1}{625}\beta^4u_x^5$$

which is part of hierarchy **VIII**, namely

$$u_t = \left(R_8^3[u] + R_8^2[u]C_2 + R_8[u]C_1 + C_0\right)\left(u_{xx} + h_8u_x^2\right),$$

with $h_8 = \beta/5$ (β is an arbitrary but nonzero constant) and $C_2 = C_1 = C_0 = 0$. This equation linearises to

$$U_T = U_{XXXXX}$$

by the transformation

$$x = U, \quad dt = dT, \quad u(x,t) = \frac{5}{\beta}\ln|U_X|.$$

3.3 Higher-order potential equations and more linearisable hierarchies

As in the case of the second-order equations (I) - (VIII) discussed in Section 2, the corresponding hierarchies (I) - (VIII) may also be written in potential form, resulting in new forms of linearisable higher-order equations. Recursion operators are presented here that generate these hierarchies of linearisable equations for (2.3), (2.9), (2.22), (2.25), (2.28), (2.31), (2.35), and (2.38). First we consider one example in detail and then list some other cases.

Detailed example for hierarchy (I): Consider the third-order equation of hierarchy (I), i.e.,

$$u_t = h_3^{3/2}\left(u_{xxx} + 3\frac{\ddot{h}_3}{\dot{h}_3}u_xu_{xx} + \frac{\dddot{h}_3}{\dot{h}_3}u_x^3\right) + C\left(h_3u_{xx} + \{h_3\}_u u_x^2\right). \tag{3.12}$$

Let

$$\begin{aligned} v_x &= h_3^{-1/2}(u) + \phi_1(x) \\ v_t &= -\frac{1}{2}\left(\dot{h}_3(u)u_{xx} + \ddot{h}_3(u)u_x^2 + Ch_3^{-1/2}\dot{h}_3u_x\right) - \lambda_1. \end{aligned} \tag{3.13}$$

The potential equation takes the form

$$v_t = \frac{v_{xxx} - \phi_{1xx}(x)}{(v_x - \phi_1(x))^3} - 3\frac{(v_{xx} - \phi_{1x})^2}{(v_x - \phi_1(x))^4} + \frac{v_{xx} - \phi_{1x}(x)}{(v_x - \phi_1(x))^2} - \lambda_1. \tag{3.14}$$

Transforming (3.14) by the pure hodograph transformation

$$v(x,t) = \chi, \quad t = \tau, \quad x = V(\chi,\tau) \tag{3.15}$$

leads to the autonomous equation

$$V_\tau = \frac{V_{\chi\chi\chi} + \phi_1'' V_\chi^4}{(1-\phi_1 V_\chi)^3} + 3\,\frac{\phi_1 V_{\chi\chi}^2 + 2\phi_1' V_\chi^2 V_{\chi\chi} + (\phi_1')^2 V_\chi^5}{(1-\phi_1 V_\chi)^4} + C\frac{V_{\chi\chi} + \phi_1' V_\chi^3}{(1-\phi_1 V_\chi)^2} + \lambda_1 V_\chi, \tag{3.16}$$

where $\phi_1' = d\phi_1(V)/dV$. Equation (3.16) linearises to

$$U_T = U_{XXX} + CU_{XX} + \lambda_1 U_X$$

by the transformation

$$V(\chi,\tau) = U(X,T), \quad d\tau = dT, \quad V_\chi^{-1} - \phi_1(V) = U_X^{-1}.$$

This is transformation (2.4), i.e., the same transformation that linearises (2.3), that is

$$V_\tau = \frac{V_{\chi\chi} + \phi_1'(V) V_\chi^3}{(1-\phi_1(V)V_\chi)^2} + \lambda_1 V_\chi.$$

Calculating the recursion operator for (2.3) we obtain

$$\tilde{R}_1[V] = \frac{1}{1-\phi_1(V)V_\chi} D_\chi + \frac{\phi_1(V)V_{\chi\chi} + \phi_1'(V)V_\chi^2}{(1-\phi_1(V)V_\chi)^2}, \tag{3.17}$$

so that

$$V_\tau = \left(\tilde{R}_1^2[V] + C\tilde{R}_1[V] + \lambda_1\right) V_\chi \tag{3.18}$$

leads to (3.16). A new hierarchy of linearisable equations is therefore given by

$$V_\tau = \left(\tilde{R}_1^m[V] + \sum_{i=0}^{m-1} \tilde{R}_1^i[V] C_i\right) V_\chi. \tag{3.19}$$

The hierarchy (3.19) is linearised by the transformation (2.4).

We list below the recursion operators for equations (2.9), (2.22), (2.25), (2.28), (2.31), (2.35), and (2.38).

Equation (2.9) of (II): The recursion operator is

$$\tilde{R}_2[V] = \frac{1}{1-\phi_2(V)V_\chi} D_\chi + \frac{\phi_2(V)V_{\chi\chi} + \phi_2'(V)V_\chi^2}{(1-\phi_2(V)V_\chi)^2} + \lambda\frac{V_\chi}{1-\phi_2(V)V_\chi} \tag{3.20}$$

and the hierarchy

$$V_\tau = \left(\tilde{R}_2^m[V] + \sum_{i=0}^{m-1} \tilde{R}_2^i[V] C_i\right) V_\chi$$

is linearisable by the transformation (2.10).

Equation (2.22) of (IV.1): The recursion operator is

$$\tilde{R}_{4.1}[V] = \frac{1}{V_\chi} D_\chi - \frac{V_{\chi\chi}}{V_\chi^2} - r \tag{3.21}$$

and the hierarchy

$$V_\tau = \left(\tilde{R}_{4.1}^m[V] + \sum_{i=0}^{m-1} \tilde{R}_{4.1}^i[V] C_i \right) \left(\frac{V_{\chi\chi}}{V_\chi^2} + \phi_4'(V) V_\chi + (2r - \lambda_1)(1 - \phi_4(V) V_\chi) \right) \tag{3.22}$$

is linearisable by the transformation (2.23). Here r is given by (2.20).

Equation (2.25) of (IV.2): The recursion operator is

$$\tilde{R}_{4.2}[V] = \frac{1}{V_\chi} D_\chi - \frac{V_{\chi\chi}}{V_\chi^2} - \lambda_1 \tag{3.23}$$

and the hierarchy

$$V_\tau = \left(\tilde{R}_{4.2}^m[V] + \sum_{i=0}^{m-1} \tilde{R}_{4.2}^i[V] C_i \right) \left(\frac{V_{\chi\chi}}{V_\chi^2} + \phi_4'(V) V_\chi + \lambda_1 (1 - \phi_4(V) V_\chi) - \frac{1}{\lambda_1} e^{\lambda_1 V} V_\chi \right) \tag{3.24}$$

is linearisable by the transformation (2.26).

Equation (2.28) of (V): The recursion operator has the same form as (3.20), namely

$$\tilde{R}_5[V] = \frac{1}{1 - \phi_5(V) V_\chi} D_\chi + \frac{\phi_5(V) V_{\chi\chi} + \phi_5'(V) V_\chi^2}{(1 - \phi_5(V) V_\chi)^2} + \lambda \frac{V_\chi}{1 - \phi_5(V) V_\chi} \tag{3.25}$$

and the hierarchy

$$V_\tau = \left(\tilde{R}_5^m[V] + \sum_{i=0}^{m-1} \tilde{R}_5^i[V] C_i \right) \left(\frac{V_{\chi\chi} + \phi_5'(V) V_\chi^3}{(1 - \phi_5(V) V_\chi)^2} + \frac{\lambda V_\chi^2}{1 - \phi_5 V_\chi} + \frac{\lambda_2}{\lambda} (1 - \phi_5(V) V_\chi) \right)$$

is linearisable by the transformation (2.29).

Equation (2.31) of (VI): The recursion operator is

$$\tilde{R}_6[V] = \frac{1}{V_\chi} D_\chi - \frac{V_{\chi\chi}}{V_\chi^2} + \frac{1}{2} \left(\frac{1 - \phi_6 V_\chi}{V_\chi} \right) \tag{3.26}$$

and the hierarchy

$$V_\tau = \left(\tilde{R}_6^m[V] + \sum_{i=0}^{m-1} \tilde{R}_6^i[V] C_i \right) \left(\frac{V_{\chi\chi}}{V_\chi^2} + \phi_6'(V) V_\chi - \frac{1}{2} \frac{(1 - \phi_6(V) V_\chi)^2}{V_\chi} \right), \tag{3.27}$$

is linearisable by the transformation (2.33).

Equation (2.35) of (VII): The recursion operator has the same form as (3.17), namely

$$\tilde{R}_7[V] = \frac{1}{1-\phi_7(V)V_\chi} D_\chi + \frac{\phi_7(V)V_{\chi\chi} + \phi_7'(V)V_\chi^2}{(1-\phi_7(V)V_\chi)^2} \tag{3.28}$$

and the hierarchy

$$V_\tau = \left(\tilde{R}_7^m[V] + \sum_{i=0}^{m-1} \tilde{R}_7^i[V] C_i \right) \left(\frac{V_{\chi\chi} + \phi_7'(V)V_\chi^3}{(1-\phi_7(V)V_\chi)^2} + \lambda_3(1-\phi_7(V)V_\chi) \right)$$

is linearisable by the transformation (2.36).

Equation (2.38) of (VIII): The recursion operator is of the same form as (3.23), namely

$$\tilde{R}_8[V] = \frac{1}{V_\chi} D_\chi - \frac{V_{\chi\chi}}{V_\chi^2} - \lambda_1 \tag{3.29}$$

and the hierarchy

$$V_\tau = \left(\tilde{R}_8^m[V] + \sum_{i=0}^{m-1} \tilde{R}_8^i[V] C_i \right) \left(\frac{V_{\chi\chi}}{V_\chi^2} + \phi_8'(V)V_\chi + \lambda_1(1-\phi_8(V)V_\chi) \right) \tag{3.30}$$

is linearisable by the transformation (2.39).

Remark: For (2.15) and (2.18) of **(III)** *we were not able to find recursion operators. The operators are not of a similar form as those given above.*

3.4 Autohodograph transformations

In [6] we introduced an autohodograph transformation for the hierarchy

$$u_t = R^m[u](u^{-2}u_x)_x, \tag{3.31}$$

where

$$R[u] = D_x^2 u^{-1} D_x^{-1} \equiv u^{-1}D_x - 2u^{-2}u_x - (u^{-2}u_{xx} - 2u^{-3}u_x^2)D_x^{-1}. \tag{3.32}$$

An autohodograph transformation is an x-generalised hodograph transformation that keeps the equation invariant. For the hierarchy (3.31) the autohodograph tranformation is

$$\begin{cases} dX(x,t) = xdx + \left\{ xD_x^{-1}R^m[u]\left(u^{-2}u_x\right)_x + \left(u^{-1}D_x\right)^m u^{-1} \right\} dt \\ dT(x,t) = dt \\ U(X,T) = x^{-1}, \end{cases} \tag{3.33}$$

The hierarchy (3.31) is a special case of hierarchy (I), with $h_1(u) = u^{-2}$. The result given in [6] can be generalised to the hierarchy (I), with arbitrary h_1 and R_1 given in subsection 3.1: The hierarchy of evolution equations

$$u_t = R_1^m[u]\left(h_1 u_{xx} + \{h_1\}_u u_x^2\right), \quad m = 1, 2, 3, \dots, \tag{3.34}$$

admits the autohodograph transformation

$$\begin{cases} dX_0(x,t) = x h_1^{-1/2} dx + \left[-\frac{1}{2} x D_x^{-1} h_1^{-3/2} \dot{h}_1 R_1^m[u]\left(h_1 u_{xx} + \{h_1\}_u u_x^2\right) \right. \\ \qquad\qquad \left. + \frac{1}{2} D_x^{-2} h_1^{-3/2} \dot{h}_1 R_1^m[u]\left(h_1 u_{xx} + \{h_1\}_u u_x^2\right) \right] dt \\ dT_0(x,t) = dt \\ U_0(x_0, t_0) = x^{-1}, \end{cases} \tag{3.35}$$

thereby transforming (3.34) to

$$U_{0T_0} = R_1^m[U_0]\left(h_1 U_{0X_0X_0} + \{h_1\}_{U_0} U_{0X_0}^2\right), \quad m = 1, 2, 3, \dots \ . \tag{3.36}$$

3.5 On the x-dependent linear equation

In our classification of equations (I) - (VIII) we use as starting point the autonomous linear equation (1.9). Alternatively we could allow the λ's to be x-dependent continuous functions. In such case one applies the pure hodograph transformation followed by the x-generalised hodograph transformation, in order to construct linearisable autonomous evolution equations. The same result then follows, i.e., equations (I) - (VIII), with only one more additional autonomous equation, namely the equation obtained from the linear equation after transforming by the pure hodograph transformation.

Consider

$$U_T = \lambda_0(X) + \sum_{k=1}^{n} \lambda_k(X) U_{X^k}. \tag{3.37}$$

Transforming the equation by the pure hodograph transformation

$$X = \omega(\xi, \eta), \quad U(X,T) = \xi, \quad T = \eta,$$

leads to the nonlinear evolution equation

$$\omega_\eta = -\lambda_0(\omega)\omega_\xi - \sum_{j=0}^{n} \lambda_{j+1}(\omega) \left(D_\xi \omega_\xi^{-1}\right)^j, \tag{3.38}$$

that is

$$\omega_\eta = -\lambda_0(\omega)\omega_\xi - \lambda_1(\omega) + \lambda_2(\omega)\omega_\xi^{-2}\omega_{\xi\xi} - \lambda_3(\omega)\left(3\omega_\xi^{-4}\omega_{\xi\xi}^2 - \omega_\xi^{-4}\omega_{\xi\xi\xi}\right) - \cdots \ .$$

Using the x-generalised hodograph transformation to transform (3.38) leads to the same hierarchies (I) - (VIII), besides classes of explicitly x-dependent evolution equations.

Acknowledgements

ME acknowledges financial support from the *Knut and Alice Wallenberg Foundation* under grant Dnr. KAW 2000.0048.

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